# Method of Multiple Scales and Identification of Nonlinear Structural Dynamic Systems

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**A procedure is developed to identify the parameters of a nonlinear structural dynamic system with a single degree of freedom. A cubic nonlinearity is assumed for purposes of illustration. In comparison to the direct identification procedures, which depend on either the availability of data on all four variables, namely, velocity, acceleration, displacement, and the input of the system, or the formulation of an algorithm that is used to numerically integrate differential equations at each iterative step, the developed procedure requires the data on only one of the field variables and no numerical integration at each step. The input to the system is also treated as an unknown. The results from the perturbation identification procedure are compared with the results from two direct identification procedures.**

## Introduction

ALTHOUGH the subject of the identification of structural dynamic systems is relatively new, there is considerable activity in this field, most of which is restricted to the consideration of linear structural dynamic systems. Identification of structural dynamic systems whose behavior can be represented more accurately by incorporating some type of nonlinearity, has not received adequate attention. This is not surprising since the inclusion of nonlinearities makes the system less amenable to modeling, analysis, and identification. However, most of the physical systems encountered in practice, are, in fact, nonlinear in nature. Even systems that can be represented by linear models tend to exhibit nonlinear behavior under certain conditions. In such cases, an engineer very often is forced to make a decision concerning the inclusion of nonlinear effects.

For example, such a problem is encountered in the analyses of the dynamic response of helicopter airframes. In these cases, the dynamic responses that have been calculated by using finite element models such as NASTRAN do not always agree with the experimental results.[1] Even the agreements of some of the higher order natural frequencies and mode shapes are not satisfactory.[1,2] Assumptions and techniques that have been used in the linear finite element models have been attributed to be possible causes for the discrepancy. Another cause of the discrepancy may be the existence of nonlinear effects, under certain conditions.

In order to include any nonlinear effects that may be present, or to rule out the existence of any nonlinear effects in such structures, there is a definite need to develop identification techniques that are applicable to nonlinear structural dynamic systems. Previous efforts in this field include the works of Ibanez,[3] who has used a describing function method for a harmonic input, and Broersen,[4] who has replaced nonlinear terms in the equations with a series expansion approximation for a system subjected to random excitation. In the latter case, the coefficients of the series expansion have been determined by using correlation techniques. This method is an extension of the statistical linearization concept. Correlation techniques have also been used by Marmarelis and Udwadia[5] to estimate the first- and higher-order kernels appearing in the Volterra series description of nonlinear systems. This approach differs from the others in that the impulse response functions of various orders are determined, and thus it represents a nonparametric method. Distefano and Rath[6] have described several methods of identification. One is a direct equation error approach. The other techniques are based on the state variable description of the system with an augmented state vector containing the unknown parameters as additional state variables. Yun and Shinozuka[7] have also used such a description of the system. They have applied nonlinear Kalman filtering techniques for estimation of the augmented state vector. McNiven and Matzen[8] have employed the Gauss-Newton method to minimize an integral of the weighted squared output error function to determine the unknown parameters.

On the basis of the experience that has been gained in the identification of linear structural dynamic systems, it is evident that the identification techniques that have relied on the nature of the analytical solutions to the system are computationally efficient. It is difficult to obtain analytical solutions for nonlinear systems. However, there are many practical systems in which the nonlinear effects are small. In such cases, approximate analytical solutions can be sought to a specific order of accuracy by using techniques such as the method of multiple scales[9] or the Krylov and Bogoliubov[10-12] method. In this paper, an identification procedure has been presented to nonlinear systems by using perturbation solutions.

A damped single-degree-of-freedom system with cubic nonlinearity has been used to exemplify the approach. Analytical solutions have been first obtained by using perturbation procedures. An input of a specified impulse has been assumed. Using these solutions, parameter identification techniques have been developed. The results from these identification procedures, which can be called perturbation identification procedures, have been compared with two other identification procedures used for nonlinear systems.

## Analytical Solution by Perturbation Methods

A nonlinear single-degree-of-freedom system is assumed in the following form:

$$\ddot{u} + 2\gamma\dot{u} + \Omega^2 u + \epsilon\alpha u^3 = f \tag{1}$$

where $f$ is the forcing function; $u(t)$ the displacement; and $\Omega$, $\gamma$, and $\epsilon\alpha$ are constants. The quantity $\epsilon$ has been assumed to be small. The damping coefficient has been assumed to be equal

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to

$$\gamma = \mu\epsilon \tag{2}$$

The dots in Eq. (1) denote differentiation with respect to time, $t$. The forcing function has been assumed to be an impulse applied at $t=0$. Thus,

$$f = Q\delta(t) \tag{3}$$

where $\delta(t)$ is the Dirac delta function and $Q$ the strength of the impulse. In identification experiments, an impulse forcing function can be simulated by using a calibrated hammer. The method can be extended to consider other forcing functions. The solution of Eq. (1) with homogeneous initial conditions and a Dirac delta forcing function is equivalent to the solution of the following homogeneous differential equation:

$$\ddot{u} + 2\mu\epsilon\dot{u} + \Omega^2 u + \epsilon\alpha u^3 = 0 \tag{4}$$

and inhomogeneous initial conditions

$$u(t=0^+) = 0; \qquad \dot{u}(t=0^+) = Q \tag{5}$$

**Method of Multiple Scales**

A solution by the method of multiple scales can be obtained by introducing two scales $T_0 = t$ and $T_1 = \epsilon t$. Thus,

$$u(t) = u(T_0, T_1) = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \epsilon^2 u_2(T_0, T_1) \tag{6}$$

By differentiating Eq. (6), substituting and equating terms of equal powers of $\epsilon$, the following equations are obtained to the order of $\epsilon^0$ and $\epsilon$.

$$\frac{\partial^2 u_0}{\partial T_0^2} + \Omega^2 u_0 = 0 \tag{7}$$

$$\frac{\partial^2 u_1}{\partial T_0^2} + \Omega^2 u_1 = -2\frac{\partial^2 u_0}{\partial T_0 \partial T_1} - 2\mu\frac{\partial u_0}{\partial T_0} - \alpha u_0^3 \tag{8}$$

A solution of Eq. (7) is

$$u_0 = A(T_1)e^{i\Omega T_0} + \bar{A}(T_1)e^{-i\Omega T_0} \tag{9}$$

where $\bar{A}(T_1)$ is the complex conjugate of $A(T_1)$. Substituting Eq. (9) into the right-hand side of Eq. (8) and simplifying, the following equation is obtained:

$$\frac{\partial^2 u_1}{\partial T_0^2} + \Omega^2 u_1 = -2\left[i\Omega(A' + \mu A) + \frac{3}{2}\alpha A^2 \bar{A}\right]e^{i\Omega T_0}$$

$$+ 2\left[i\Omega(\bar{A}' + \mu\bar{A}) - \frac{3}{2}\alpha A^2 \bar{A}\right]e^{-i\Omega T_0}$$

$$-\alpha[A^3 e^{3i\Omega T_0} + \bar{A}^3 e^{-3i\Omega T_0}] \tag{10}$$

where the prime represents differentiation with respect to the argument $T_1$. The secular terms in solution (10) are eliminated by setting

$$2i\Omega(A' + \mu A) + 3\alpha A^2 \bar{A} = 0 \tag{11}$$

$$2i\Omega(\bar{A}' + \mu\bar{A}) - 3\alpha A^2 \bar{A} = 0 \tag{12}$$

Solutions to these equations are obtained by setting

$$A = (a/2)e^{i\beta} \tag{13}$$

where $a$ and $\beta$ are real functions of $T_1$. By substituting Eq. (13) in (11) and simplifying

$$a' + \mu a = 0 \tag{14}$$

$$-\beta' + (3\alpha a^2/8\Omega) = 0 \tag{15}$$

or

$$a = a_0 e^{-\mu T_1} \tag{16}$$

$$\beta = (3\alpha a_0^2/16\mu\Omega)(1 - e^{-2\mu T_1}) + \beta_0 \tag{17}$$

The constants $a_0$ and $\beta_0$ are obtained from initial conditions. By substituting for $T_0$ and $T_1$ in terms of $t$, the general solution for $u_0(t)$ can be written as follows:

$$u(t) = a_0 e^{-\mu\epsilon t}\cos[\Omega t + (3\alpha a_0^2/16\Omega\mu)(1 - e^{-2\mu\epsilon t}) + \beta_0] \tag{18}$$

From the initial conditions

$$\beta_0 = \pi/2 \tag{19}$$

and $a_0$ is obtained by solving the cubic equation

$$a_0^3 + (8\Omega^2/3\alpha\epsilon)a_0 + (8\Omega/3\alpha\epsilon)Q = 0 \tag{20}$$

## Parameter Identification

The perturbation solutions developed in the preceding sections are used as the basis for identifying the system parameters $\gamma$, $\epsilon$, and $\Omega^2$. In addition, the impulse can also be treated as an unknown parameter. An integral least-squares approach is employed to define an objective function and minimize it with respect to the unknown parameters. The experimental data required for identification can be either the displacement, velocity, or acceleration time history. Equation (18) and its derivatives with respect to time are used as the analytical counterparts which are fitted to the measured time histories.

The objective function to be minimized is written as

$$L(\theta) = \int_0^T (\bar{u}_m - \bar{u}_a)^2 dt \tag{21}$$

where $\bar{u}_m$ and $\bar{u}_a$, respectively, denote the measured and analytical displacement, velocity or acceleration response. The quantity $T$ is the record length and $\{\theta\}$ the vector of unknown parameters.

$$\{\theta\} = \begin{Bmatrix} \gamma \\ \Omega^2 \\ \epsilon\alpha \\ Q \end{Bmatrix} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} \tag{22}$$

Minimization of the function $L(\theta)$ is accomplished by using the Levenberg-Marquardt method[13,14] which can be viewed as a modification of the Gauss-Newton method. It is iterative in nature, and starting values for all of the unknown parameters must be provided.

If $\{\theta^i\}$ denotes the estimate of $\{\theta\}$ after the $i$th iteration, $\{\theta^{i+1}\}$ is obtained as

$$\{\theta^{i+1}\} = \{\theta^i\} + h^i\{\Delta\theta^i\} \tag{23}$$

where $\{\Delta\theta^i\}$ is the correction vector computed during the $(i+1)$th iteration and $h^i$ is the step size. The correction vector is determined by solving the following set of simultaneous equations:

$$\left[ [(N^i)]^* + \lambda^i [I] \right] \{ (\Delta\theta^i)^* \} = \{ (q^i)^* \} \tag{24}$$

where

$$(q^i)^*_k = q^i_k / \sqrt{N^i_{kk}} \tag{25}$$

$$(N^i)^*_{kl} = N^i_{kl} / \sqrt{N^i_{kk} N^i_{ll}} \tag{26}$$

$$(\Delta\theta^i)^*_k = (\Delta\theta^i_k)(\sqrt{N^i_{kk}}) \tag{27}$$

$$q^i_k = -2 \int_0^T \left( \frac{\partial \bar{u}^i_a}{\partial \theta_k} \right) (\bar{u}_m - \bar{u}^i_a) \mathrm{d}t \tag{28a}$$

and

$$N^i_{kl} = 2 \int_0^T \left( \frac{\partial \bar{u}^i_a}{\partial \theta_k} \right) \left( \frac{\partial \bar{u}^i_a}{\partial \theta_l} \right) \mathrm{d}t \tag{28b}$$

The quantities $\bar{u}^i_a$ and $(\partial \bar{u}^i_a / \partial \theta_k)$ represent $\bar{u}_a$ and $(\partial \bar{u}_a / \partial \theta_k)$ evaluated at $\{\theta\} = \{\theta^i\}$. The quantity $\{q\}$ represents the gradient vector and $[N]$ is the approximation to the Hessian or the second derivative matrix.

The identity matrix with proper dimensions is denoted by $(I)$. The quantity $\lambda^i$ is a constant, sometimes referred to as the Marquardt parameter. $[N^*]$, $\{q^*\}$, and $\{\Delta\theta^*\}$ are used to denote scaled versions of $[N]$, $\{q\}$, and $\{\Delta\theta\}$, respectively. The scaling is done to improve the accuracy of numerical computations. The effect of scaling is such that all of the diagonal elements of $[N]$ are replaced by unity. The presence of the term $\lambda^i [I]$ in Eq. (23) ensures a solution for this equation even in cases where $[N]$ is extremely ill-conditioned. Selection of the parameters $h^i$ and $\lambda^i$ is made following a procedure outlined by Bard.[15] The constant $\lambda^i$ is set equal to 0.01 for the first iteration. This value is changed by a factor of 10 during successive iterations in accordance with the objective function value at $\{\theta\} = \{\bar{\theta}^i\}$, where

$$\{\bar{\theta}^i\} = \{\theta^i\} + \{\Delta\theta^i\} \tag{29}$$

The constant $\lambda^i$ is multiplied by ten if $L(\bar{\theta}^i) \geq L(\theta^i)$ and $\lambda^i$ is divided by ten otherwise to obtain $\lambda^{i+1}$.

The technique for choosing $h^i$ involves a quadratic interpolation scheme that consists of expressing the objective function $L(\theta)$ as a quadratic in the step-size parameter and computing the minimum of this function. Other details of this procedure are discussed in Ref. 15.

A termination of the estimation procedure, after the required accuracy has been achieved, is accomplished according to the following convergence criterion:

$$(\theta^{i+1}_k - \theta^i_k)/\theta^i_k \leq 0.0005 \tag{30}$$

for all $k$; that is, the parameters are assumed to have converged if the relative changes in their values in two successive iterations are less than 0.05%.

## Noniterative Direct Identification

As described in the preceding sections, the perturbation approach to structural dynamic system identification is based on an analytical solution that is valid for small values of $\epsilon$. Only a measurement of one of the quantities such as , $u$, $\dot{u}$, or $\ddot{u}$ is required. In this paper, the results from the perturbation procedure have been compared with a noniterative direct identification procedure that is applicable to nonlinear equations of this type. Such a procedure has been discussed in Ref. 6. In this noniterative approach, measurements of quantities $u$, $\dot{u}$, $\ddot{u}$ are required. In addition to these measurements, it is also necessary to measure the magnitude of the impulse $Q$. It should be noted that the magnitude of $Q$ need not be measured in the perturbation identification procedure. The magnitude of the impulse is identified along with the other parameters, $\gamma$, $\Omega^2$, and $\epsilon\alpha$.

As applied to Eq. (1) the direct method consists of defining an error function

$$L(\theta) = \int_0^T (\ddot{u}_m + 2\gamma \dot{u}_m + \Omega^2 u_m + \epsilon\alpha u_m^3 - f)^2 \mathrm{d}t \tag{31}$$

where $T$ is the record length. The subscript $m$ denotes the measured quantities. The parameters are obtained by minimizing $L(\theta)$.

## Iterative Direct Identification

The developed perturbation identification method has also been compared with an iterative direct identification procedure that is described in Ref. 6 and is applicable to nonlinear systems. This method can be briefly explained as follows. Instead of using the perturbation solutions to compute $\bar{u}_a$ and $\theta_k$, these quantities are obtained by numerical integration. Specifically, Eq. (1) can be rewritten in terms of $\{\theta\}$.

$$\ddot{u} + 2\theta_1 \dot{u} + \theta_2 u + \theta_3 u^3 = \theta_4 \delta(t) \tag{32}$$

Then, the following variables are defined:

$$y_1 = u; \quad y_2 = u,_t; \quad y_3 = u,_{\theta_1}; \quad y_4 = (u,_{\theta_1}),_t$$

$$y_5 = u,_{\theta_2}; \quad y_6 = (u,_{\theta_2}),_t; \quad y_7 = u,_{\theta_3}$$

$$y_8 = (u,_{\theta_3}),_t; \quad y_9 = u,_{\theta_4}; \quad y_{10} = (u,_{\theta_4}),_t \tag{33}$$

**Table 1 Effect of nonlinear parameter (perturbation approach)**

| $\epsilon\alpha$ | Noise level | Estimates | | | |
|---|---|---|---|---|---|
| | | $\gamma$ | $\Omega^2$ | $\epsilon\alpha$ | $Q$ |
| | Initial | 0.35 | 22.50 | 1.00 | 3.50 |
| 0.5 | 0% | 0.50 | 24.76 | 0.47 | 5.01 |
| | 5% | 0.50 | 24.77 | 0.47 | 4.99 |
| | 10% | 0.50 | 24.77 | 0.47 | 4.98 |
| | 20% | 0.49 | 24.78 | 0.45 | 4.94 |
| | Exact | 0.50 | 25.00 | 0.50 | 5.00 |
| | Initial | 0.35 | 22.50 | 0.50 | 3.50 |
| 1.5 | 0% | 0.50 | 24.78 | 1.40 | 5.02 |
| | 5% | 0.49 | 24.78 | 1.42 | 5.01 |
| | 10% | 0.49 | 24.78 | 1.44 | 4.99 |
| | 20% | 0.49 | 24.78 | 1.58 | 4.96 |
| | Exact | 0.50 | 25.00 | 1.50 | 5.00 |
| | Initial | 0.35 | 22.50 | 1.50 | 3.50 |
| 2.5 | 0% | 0.49 | 24.80 | 2.34 | 5.04 |
| | 5% | 0.49 | 24.79 | 2.38 | 5.02 |
| | 10% | 0.49 | 24.79 | 2.42 | 5.01 |
| | 20% | 0.49 | 24.78 | 2.51 | 4.97 |
| | Exact | 0.50 | 25.00 | 2.50 | 5.00 |
| | Initial | 0.35 | 22.50 | 2.50 | 3.50 |
| 5.0 | 0% | 0.48 | 24.83 | 4.69 | 5.07 |
| | 5% | 0.48 | 24.82 | 4.79 | 5.05 |
| | 10% | 0.48 | 24.80 | 4.89 | 5.04 |
| | 20% | 0.48 | 24.78 | 5.10 | 5.00 |
| | Exact | 0.50 | 25.00 | 5.00 | 5.00 |

Table 2 Effect of record length (perturbation approach)

| $T$, s | Noise level | Estimates $\gamma$ | $\Omega^2$ | $\epsilon\alpha$ | $Q$ |
|---|---|---|---|---|---|
| | Initial | 0.35 | 22.50 | 1.00 | 3.50 |
| 2.5 | 0% | 0.49 | 24.80 | 2.40 | 5.02 |
| | 5% | 0.48 | 24.80 | 2.48 | 5.00 |
| | 10% | 0.48 | 24.79 | 2.56 | 4.98 |
| | 20% | 0.47 | 24.78 | 2.74 | 4.93 |
| 5.0 | 0% | 0.49 | 24.80 | 2.34 | 5.04 |
| | 5% | 0.49 | 24.79 | 2.38 | 5.02 |
| | 10% | 0.49 | 24.79 | 2.42 | 5.01 |
| | 20% | 0.49 | 24.78 | 2.51 | 4.97 |
| 7.5 | 0% | 0.49 | 24.79 | 2.34 | 5.04 |
| | 5% | 0.49 | 24.79 | 2.38 | 5.03 |
| | 10% | 0.49 | 24.79 | 2.42 | 5.01 |
| | 20% | 0.49 | 24.78 | 2.51 | 4.98 |
| | Exact | 0.50 | 25.00 | 2.50 | 5.00 |

By differentiating Eq. (32) with respect to $\theta_k$ and representing the effect of the impulsive force as initial conditions for $t=0^+$, the following set of first-order differential equations can be written:

$$\dot{y}_1 = y_2 \tag{34}$$

$$\dot{y}_2 = -2\theta_1 y_2 - (\theta_2 + \theta_3 y_1^2) y_1 \tag{35}$$

$$\dot{y}_3 = y_4 \tag{36}$$

$$\dot{y}_4 = -2y_2 - 2\theta_1 y_4 - (\theta_2 + 3\theta_3 y_1^2) y_3 \tag{37}$$

$$\dot{y}_5 = y_6 \tag{38}$$

$$\dot{y}_6 = -y_1 - 2\theta_1 y_6 - (\theta_2 + 3\theta_3 y_1^2) y_5 \tag{39}$$

$$\dot{y}_7 = y_8 \tag{40}$$

$$\dot{y}_8 = -y_1^3 - 2\theta_1 y_8 - (\theta_2 + 3\theta_3 y_1^2) y_7 \tag{41}$$

$$\dot{y}_9 = y_{10} \tag{42}$$

$$\dot{y}_{10} = -2\theta_1 y_{10} - (\theta_2 + 3\theta_3 y_1^2) y_9 \tag{43}$$

The initial conditions for this set of equations are

$$y_1 = y_3 = y_4 = y_5 = y_6 = y_7 = y_8 = y_9 = 0 \tag{44}$$

$$y_2 = \theta_4, \quad y_{10} = 1 \tag{45}$$

Equations (34-45) are used in the identification algorithm. During each iteration, the vector $\{\theta\}$ is selected. Then Eqs. (34-45) are solved by a numerical procedure to provide the displacement, velocity, acceleration time histories, and derivatives with respect to the parameters needed in Eq. (21). As $\{\theta\}$ changes, it is necessary to implement the numerical procedure again.

## Numerical Results

The performance of the perturbation identification procedure has been evaluated by considering Eq. (1). This equation, however, can be written in terms of a normalized time $\tau = \Omega t$. Then,

$$\frac{d^2u}{d\tau^2} + 2\left(\frac{\gamma}{\Omega}\right)\frac{du}{d\tau} + u + \left(\frac{\epsilon\alpha}{\Omega^2}\right)u^3 = \frac{Q}{\Omega^2}\delta(\tau) \tag{46}$$

Table 3 Effect of nonlinear parameter (direct approach)

| $\epsilon\alpha$ | Noise level | Estimates $\gamma$ | $\Omega^2$ | $\epsilon\alpha$ |
|---|---|---|---|---|
| 0.5 | 0% | 0.50 | 25.00 | 0.50 |
| | 5% | 0.51 | 24.99 | 0.68 |
| | 10% | 0.52 | 25.08 | 0.51 |
| | 20% | 0.53 | 25.43 | −0.64 |
| | Exact | 0.50 | 25.00 | 0.50 |
| 1.5 | 0% | 0.50 | 25.00 | 1.50 |
| | 5% | 0.51 | 25.00 | 1.70 |
| | 10% | 0.51 | 25.10 | 1.49 |
| | 20% | 0.53 | 25.48 | 0.23 |
| | Exact | 0.50 | 25.00 | 1.50 |
| 2.5 | 0% | 0.50 | 25.00 | 2.50 |
| | 5% | 0.51 | 25.01 | 2.69 |
| | 10% | 0.51 | 25.12 | 2.46 |
| | 20% | 0.52 | 25.54 | 1.09 |
| | Exact | 0.50 | 25.00 | 2.50 |
| 5.0 | 0% | 0.50 | 25.00 | 5.00 |
| | 5% | 0.51 | 25.03 | 5.18 |
| | 10% | 0.51 | 25.17 | 4.87 |
| | 20% | 0.51 | 25.67 | 3.20 |
| | Exact | 0.50 | 25.00 | 5.00 |

For values of $\gamma = 0.5$, $\Omega = 5$, and $Q = 5$, the parameters of differential equation (46) are

$$\gamma/\Omega = 0.1, \quad Q/\Omega^2 = 0.2 \tag{47}$$

A range of values of the nonlinearity parameter

$$0.02 \leq \epsilon\alpha/\Omega^2 \leq 0.2$$

has been considered. The corresponding values of $\epsilon\alpha$ have been calculated. Before the impulse is applied, the values of quantities $u$ and $\dot{u}$ have been assumed to be zero.

First, a fourth-order Runge-Kutta procedure has been used to obtain a numerical solution for a selected value of $\epsilon\alpha$. This numerical solution has been treated as a noise-free response of the system. Next, assuming the parameters of Eq. (32) or (46) were unknown, the noise-free response and the perturbation identification procedure have been used to estimate the parameters. The impulse $Q$ has also been considered to be an unknown. In addition to the noise-free response, responses with noise have been produced from the numerical solution. This has been accomplished by corrupting the numerical solution with 5, 10, and 20% random noise. A procedure similar to that discussed in Ref. 6 has been used. The parameters were again extracted by using the perturbation identification procedure by using the noise corrupted data. A sampling interval of 0.01 has been used in all cases.

The results are summarized in Table 1 for four different values of $\epsilon\alpha$. A record length of $T = 5.0$ s has been assumed for all cases. Only acceleration response has been considered to be the data. Initial estimates are shown in the appropriate tables.

For a small value of $\epsilon\alpha = 0.5$, the quantities $\gamma$ and $\Omega^2$ have been estimated exactly. For values of noise less than 10%, the error in the estimate of $\epsilon\alpha$ is 6%. As the value of $\epsilon\alpha$ is increased the error in estimating the value of $\epsilon\alpha$ also increases in some cases, as shown in Table 1. This is due to the fact that the perturbation identification procedure depends on a solution that is only accurate to the order of $\epsilon$. For higher values of $\epsilon\alpha$, higher order solutions are necessary. However, in all cases considered, $\gamma$ and $\Omega^2$ have been estimated reasonably accurately. A comparison of the simulated acceleration data and responses by using the identified equations are shown in Figs. 1 and 2.

The effect of record length has been studied. The results are shown in Table 2 for three different record lengths. For a low noise level, as expected, the record length has very little influence. For a noise level of 20% a longer record length increases the accuracy in the estimation of $\epsilon\alpha$.

#### Comparison with Noniterative Direct Identification

As a next step, the results from the perturbation identification procedures have been compared with the results from the noniterative direct identification procedure discussed previously. In this case the value of the input impulse cannot be left as an unknown parameter. The value of the impulse should be measured. Errors in the measurements have a direct influence on the values of the estimated parameters. Furthermore, the values of $u$, $\dot{u}$, and $\ddot{u}$ are needed. Results for different values of the nonlinear parameter $\epsilon\alpha$ are shown in Table 3 for a record length of $T=5$ s. The estimates of $\epsilon\alpha$ are very sensitive to noise level. For example, at 20% noise level and a low value of $\epsilon\alpha$ the error is 228%. The perturbation procedure, however, provided an estimation with 6% error. In Table 4, the effect of record length on the parameter estimation has been illustrated. For a noise level of 20%, an increase of record length from 2.5 to 5 s decreases the error of estimation in $\epsilon\alpha$ from 63 to 53%.

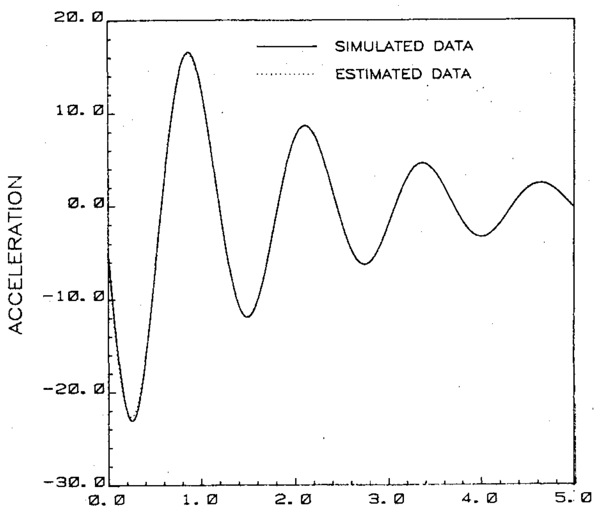


**Fig. 1 Simulated data with no noise and its fit.**

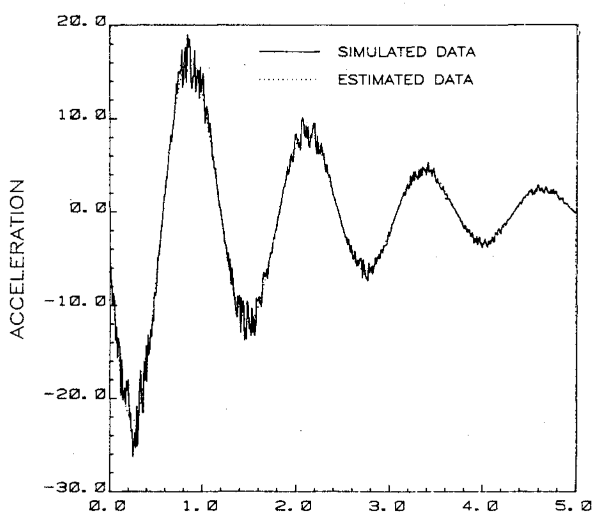


**Fig. 2 Simulated data with 20% noise and its fit.**

#### Comparison with Iterative Direct Identification

The results from the perturbation identification procedure have also been compared with the results from an iterative identification procedure. The same set of data has been used to estimate the parameters in all methods. For numerical integration inside the identification method, a fourth-order Runge-Kutta method has been used. In Table 5, the estimated

**Table 4 Effect of record length (direct approach)**

| $T$, s | Noise level | Estimates $\gamma$ | $\Omega^2$ | $\epsilon\alpha$ |
|---|---|---|---|---|
| 2.5 | 0% | 0.50 | 25.00 | 2.50 |
| | 5% | 0.51 | 24.98 | 2.74 |
| | 10% | 0.51 | 25.11 | 2.48 |
| | 20% | 0.52 | 25.64 | 0.92 |
| 5.0 | 0% | 0.50 | 25.00 | 2.50 |
| | 5% | 0.51 | 25.01 | 2.69 |
| | 10% | 0.51 | 25.12 | 2.46 |
| | 20% | 0.52 | 25.54 | 1.09 |
| 7.5 | 0% | 0.50 | 25.00 | 2.50 |
| | 5% | 0.51 | 25.00 | 2.70 |
| | 10% | 0.51 | 25.11 | 2.47 |
| | 20% | 0.52 | 25.52 | 1.12 |
| | Exact | 0.50 | 25.00 | 2.50 |

**Table 5 Effect of nonlinear parameter (iterative direct approach)**

| $\epsilon\alpha$ | Noise level | Estimates $\gamma$ | $\Omega^2$ | $\epsilon\alpha$ | $Q$ |
|---|---|---|---|---|---|
| | Initial | 0.35 | 22.50 | 1.00 | 3.50 |
| 0.5 | 0% | 0.50 | 25.00 | 0.50 | 5.00 |
| | 5% | 0.50 | 25.03 | 0.42 | 4.99 |
| | 10% | 0.50 | 25.05 | 0.34 | 4.97 |
| | 20% | 0.50 | 25.11 | 0.17 | 4.95 |
| | Exact | 0.50 | 25.00 | 0.50 | 5.00 |
| | Initial | 0.35 | 22.50 | 0.50 | 3.50 |
| 1.5 | 0% | 0.50 | 25.00 | 1.50 | 5.00 |
| | 5% | 0.50 | 25.02 | 1.44 | 4.99 |
| | 10% | 0.50 | 25.05 | 1.38 | 4.97 |
| | 20% | 0.50 | 25.09 | 1.25 | 4.94 |
| | Exact | 0.50 | 25.00 | 1.50 | 5.00 |
| | Initial | 0.35 | 22.50 | 1.50 | 3.50 |
| 2.5 | 0% | 0.50 | 25.00 | 2.50 | 5.00 |
| | 5% | 0.50 | 25.02 | 2.47 | 4.99 |
| | 10% | 0.50 | 25.04 | 2.43 | 4.97 |
| | 20% | 0.50 | 25.07 | 2.36 | 4.94 |
| | Exact | 0.50 | 25.00 | 2.50 | 5.00 |
| | Initial | 0.35 | 22.50 | 2.50 | 3.50 |
| 5.0 | 0% | 0.50 | 25.00 | 5.00 | 5.00 |
| | 5% | 0.50 | 25.00 | 5.04 | 4.98 |
| | 10% | 0.50 | 25.01 | 5.08 | 4.97 |
| | 20% | 0.50 | 25.01 | 5.17 | 4.94 |
| | Exact | 0.50 | 25.00 | 5.00 | 5.00 |

**Table 6 Effect of record length (iterative direct approach)**

| $T$, s | Noise level | Estimates $\gamma$ | $\Omega^2$ | $\epsilon\alpha$ | $Q$ |
|---|---|---|---|---|---|
| | Initial | 0.35 | 22.50 | 1.50 | 3.50 |
| 2.5 | 0% | 0.50 | 25.00 | 2.50 | 5.00 |
| | 5% | 0.50 | 25.08 | 2.34 | 4.98 |
| | 10% | 0.50 | 25.15 | 2.18 | 4.97 |
| | 20% | 0.49 | 25.31 | 1.85 | 4.93 |
| 5.0 | 0% | 0.50 | 25.00 | 2.50 | 5.00 |
| | 5% | 0.50 | 25.02 | 2.47 | 4.99 |
| | 10% | 0.50 | 25.04 | 2.43 | 4.97 |
| | 20% | 0.50 | 25.07 | 2.36 | 4.94 |
| 7.5 | 0% | 0.50 | 25.00 | 2.50 | 5.00 |
| | 5% | 0.50 | 25.01 | 2.48 | 4.99 |
| | 10% | 0.50 | 25.02 | 2.46 | 4.97 |
| | 20% | 0.50 | 25.05 | 2.42 | 4.94 |
| | Exact | 0.50 | 25.00 | 2.50 | 5.00 |

parameters have been listed for a record length of 5 s, different values of exact $\epsilon\alpha$, and different noise levels.

In all cases, iterative direct methods require a large number of iterations and increasing amounts of computer time. For the noise-free cases the direct method converges to the exact values of the parameters. The situation is different in the case of data with noise. For low values of $\epsilon\alpha$ and increasing noise level, the perturbation identification procedure is superior to the iterative direct identification procedure. For a noise level of 20% and a record length of 5 s, the error in the perturbation identification procedure is 6%. The corresponding error in the iterative direct identification procedure is 66%. In Table 6, the results for increasing record length have been presented. Again, in this method increasing values of record length result in increasing accuracy. For low values of record length and increasing noise levels, the perturbation identification procedure is found to yield better estimates.

## Conclusion

A perturbation identification procedure has been developed for a dynamic system with a single degree of freedom and a cubic nonlinearity. The method provides accurate values and is superior to both iterative and noniterative direct identification procedures for small values of $\epsilon\alpha$ and high noise levels.

In comparison to the noniterative direct identification procedure, the perturbation procedure does not require a measurement of the impulse. It requires the measurement of only one quantity, such as the acceleration, velocity, or displacement. In comparison to the iterative direct identification procedure, the perturbation procedure provides an improved computational efficiency.

The method, however, is limited by the solution which is accurate to the order of $\epsilon$. For larger values of $\epsilon\alpha$, it is necessary to obtain higher order solutions or resort to the iterative direct identification procedure.

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